characteristic polynomial (cont'd)

recall: degree of a polynomial is the highest power

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

similar to a degree-n polynomial having up to n roots, $n \times n$ matrix has, at most, n real eigenvalues

if n is odd, it has, at least, one real eigenvalue

ex. find all eigenvalue(s) of

$$A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

upper triangular matrix !!

multiplicities (# times roots repeat)

$$f_A(\lambda) = (5-\lambda)^3 (4-\lambda)^2$$

$$\lambda_1 = 5 \quad \lambda_2 = 4$$

ex. find all eigenvalue(s) of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\det(A - \lambda I_3) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \right) = 0$$

$$(1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - 1\det\begin{bmatrix} 1 & 1 \\ 1 & 1-\lambda \end{bmatrix} + 1\det\begin{bmatrix} 1 & 1-\lambda \\ 1 & 1 \end{bmatrix} = 0$$

$$(1-\lambda)\left((1-\lambda)^2 - 1\right) - (\cancel{1} - \lambda - \cancel{1}) + 1 - (1-\lambda) = 0$$

$$(1-\lambda)^3 - 1 + \lambda + \lambda + \cancel{1} - \cancel{1} + \lambda = 0$$

$$(1-\lambda)^3 + 3\lambda - 1 = 0$$

$$-\lambda^3 + 3\lambda^2 - \cancel{3\lambda} + \cancel{1} + \cancel{3\lambda} - \cancel{1} = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^2(\lambda - 3) = 0 \longrightarrow f_A(\lambda) = \lambda^2(3-\lambda) = -\lambda^3 + 3\lambda^2$$

LC (−) degree is odd

$$-\lambda^3 + 3\lambda^2 = 0$$

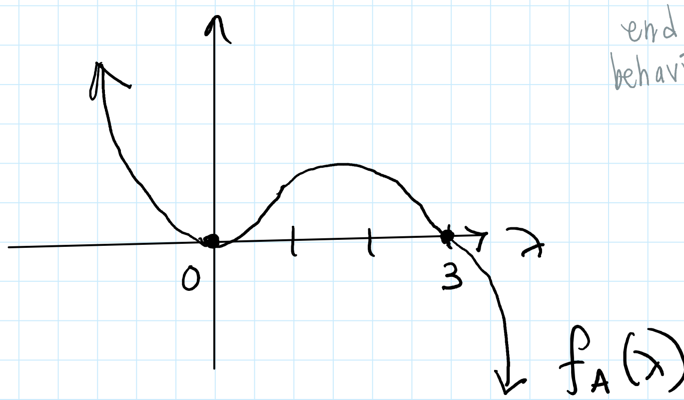
$$-\lambda^2(\lambda - 3) = 0 \longrightarrow f_A(\lambda) = \lambda^2(3-\lambda) = -\lambda^{3} + 3\lambda^2$$

end behavior

$\lambda_1 = 0 \qquad \lambda_2 = 3$

mult: $\quad 2 \qquad\qquad 1$

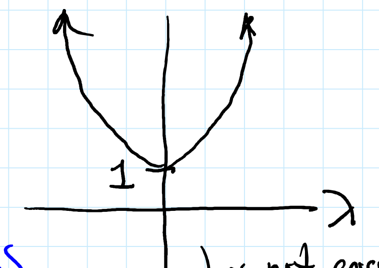
even touches $\qquad$ odd crosses



ex. given CCW 90° rotation matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$f_A(\lambda) = \det(A - \lambda I_2)$$
$$= \det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$
$$= \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

no real eigenvalues



does not cross horizontal axis

§7.3 Finding Eigenvectors of a Matrix

recall: $A\vec{v} = \lambda\vec{v} \implies (A - \lambda I_n)\vec{v} = \vec{0}$

now, focus on eigenvectors, $\vec{v} \in \mathbb{R}^n$ of matrix $A - \lambda I_n$

i.e. find $E_\lambda = \ker(A - \lambda I_n)$

eigenspace $\qquad = \vec{v} \in \mathbb{R}^n$ s.t. $A\vec{v} = \lambda\vec{v}$

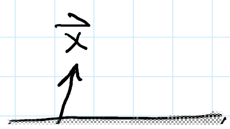
i.e. eigenspace, E_λ, contains the non-zero eigenvectors w eigenvalue(s), λ_n

ex. given $T(\vec{x}) = A\vec{x}$ is orthogonal projection onto plane $V \in \mathbb{R}^3$

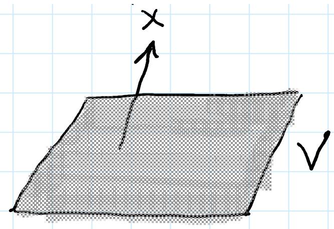
then E_1 contains solutions of $A\vec{v} = 1\vec{v}$

eigenvalue λ_1

streamline: $E_1 = V$



streamline: $E_1 = V$

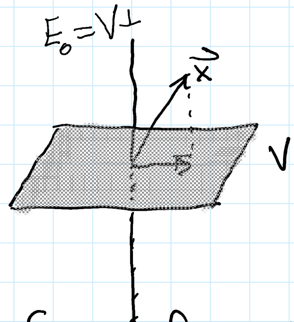


E_0 contains solutions of $A\vec{v} = 0\vec{v} = \vec{0}$, which is set of vectors perpendicular to V

$\downarrow$

streamline: $E_0 = \ker A$

$V_\perp$ is a line



revisit $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ found $\lambda_1 = 5$, $\lambda_2 = -1$ find its eigenspaces

then $E_5 = \ker(A - 5I_2) = \ker\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} = \ker\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$

$E_5 = \text{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\frac{2x_1}{2} = \frac{1}{2}x_2$

$x_2 = \text{arb. } r$

$\vec{w}_1 = \begin{bmatrix} \frac{1}{2}r \\ r \end{bmatrix} = r\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

likewise $E_{-1} = \ker(A - (-1)I_2)$

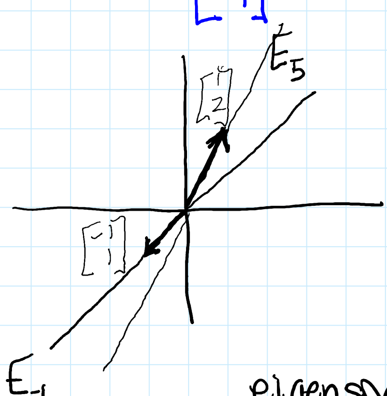
$= \ker(A + I_2) = \ker\begin{bmatrix} 1+1 & 2 \\ 4 & 3+1 \end{bmatrix}$

$E_{-1} = \text{span}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \ker\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$

$\xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2$, let $x_2 = \text{arb } t$

$\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$



eigenspaces E_5, E_{-1} are lines

A is 2×2 matrix $\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ form an eigenbasis for A

A is 2×2 matrix $\therefore \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}$ form an eigenbasis for A

can also say A is diagonalizable

$$AS = SB \quad \text{where } S = \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix} \quad \text{and } B = \begin{bmatrix}5 & 0\\0 & -1\end{bmatrix}$$

because $\lambda_1 = 5$, $\lambda_2 = -1$

ex. find eigenspaces given $A = \begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix}$ ← eigenvalues are diagonal entries

$\lambda_1 = 1 \qquad \lambda_2 = 0$

mult: $\quad 2 \qquad\quad 1$

$$E_1 = \ker(A - 1I_3) = 0 = \ker\begin{bmatrix}1-1 & 1 & 1\\0 & 0-1 & 1\\0 & 0 & 1-1\end{bmatrix}$$

$$= \ker\begin{bmatrix}0 & 1 & 1\\0 & -1 & 1\\0 & 0 & 0\end{bmatrix} \xrightarrow{rref} \ker\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$E_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$

$\left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$ $\quad x_2 = 0$, $x_3 = 0$, $x_1 = r$

$\vec{w}_1 = r\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$E_0 = \ker\begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix} \xrightarrow{rref} \ker\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$x_1 = -x_2$

$x_3 = 0$

let $x_2 =$ arb t.

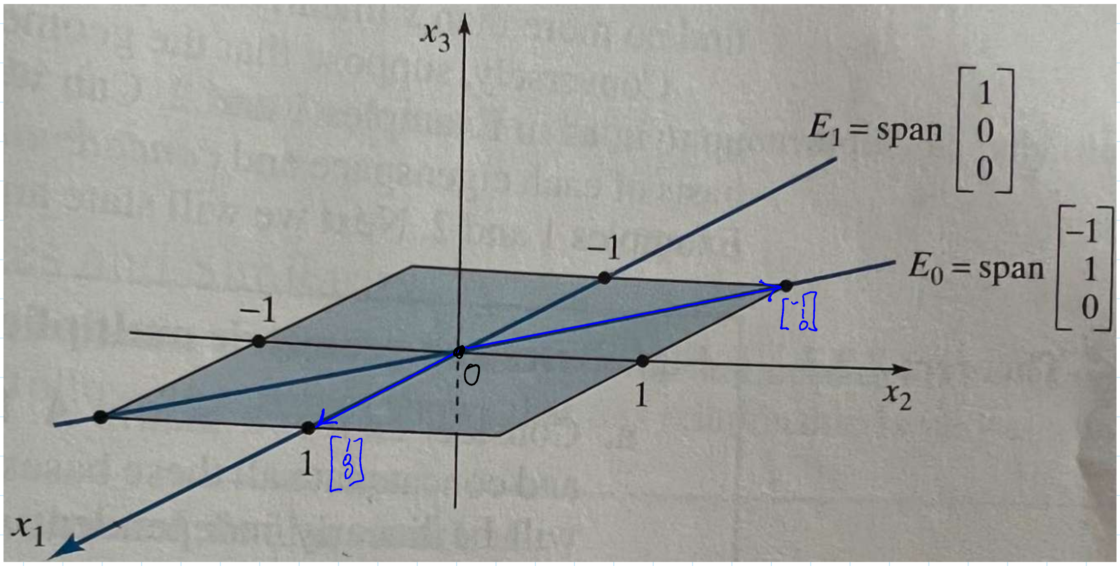
$\vec{w}_2 = \begin{bmatrix}-t\\t\\0\end{bmatrix} = t\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\text{span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$

$E_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \qquad E_0 = \text{span}\begin{bmatrix}-1\\1\\0\end{bmatrix}$

↑ both eigenspaces are lines in x_1-x_2 plane (see below)

Geometric Representation:


x_3
$E_1 = \text{span}\begin{bmatrix}1\\0\\0\end{bmatrix}$
$E_0 = \text{span}\begin{bmatrix}-1\\1\\0\end{bmatrix}$
−1
−1
0
1
1
x_2
x_1
$\begin{bmatrix}-1\\1\\0\end{bmatrix}$
$\begin{bmatrix}1\\0\\0\end{bmatrix}$